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SOME REMARKS ON BRAIDED GROUP RECONSTRUCTION AND BRAIDED DOUBLES

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ABSTRACT The cross coproduct braided group $\mathrm{Aut}(\mathcal{C}){\blacktriangleright\!\!<}B$ is obtained by Tannaka-Krein reconstruction from $\mathcal{C}^B\to\mathcal{C}$ for a braided group B in braided category $\mathcal{C}$. We apply this construction to obtain partial solutions to two problems in braided group theory, namely the tensor problem and the braided double. We obtain $\mathrm{Aut}(\mathcal{C}){\blacktriangleright\!\!<}\mathrm{Aut}(\mathcal{C})$ $\cong\mathrm{Aut}(\mathcal{C}){>\!\!\!\triangleleft}\mathrm{Aut}(\mathcal{C})$ and higher braided group 'spin chains'. The example of the braided group $B(R){>\!\!\!\triangleleft}B(R){>\!\!\!\triangleleft}\cdots{>\!\!\!\triangleleft}B(R)$ is described explicitly by R-matrix relations. We also obtain $\mathrm{Aut}(\mathcal{C}){\blacktriangleright\!\!<}\mathrm{Aut}(\mathcal{C})^*$ as a dual quasitriangular 'codouble' braided group by reconstruction from the dual category $\mathcal{C}^\circ\to\mathcal{C}$. General braided double crossproducts $B\bowtie C$ are also considered.

1 Introduction

Recently there has been a lot of interest in the braided version of the Tannaka-Krein reconstruction theorem proven by the author in [1]. Here one considers a monoidal functor $F:\mathcal{C}\to\mathcal{V}$ and reconstructs (under certain representability assumptions) a braided group or Hopf algebra in the braided category $\mathcal{V}$, denoted $\mathrm{Aut}(\mathcal{C},F,\mathcal{V})$. This is how braided groups were first introduced (in [1]). By now there is a rich and extensive theory of braided groups, see for example [2][3][4][5] [6][7][8][9][10] [11][12], and [13][14] for reviews.

In this paper we consider some examples of braided reconstruction. Let B be a braided group in a category $\mathcal{C}$ and $\mathcal{C}^B$ the category of braided comodules of B. In Section 1 we study the solution of the reconstruction problem for the forgetful functor

$$\mathcal{C}^B\to\mathcal{C},$$

namely the prebosonisation braided group cross coproduct $\mathrm{Aut}(\mathcal{C}){\blacktriangleright\!\!<}B$ introduced in [4] (there in the module version rather than comodule version but the reversal of arrows is routine). It is the prebosonisation of B because in the case $\mathcal{C}=\mathcal{M}^H$ (H dual quasitriangular) it is related by

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transmutation to the bosonisation $H{\triangleright\!\!\!<}B$. All this was known since 1991 from the bosonisation theory [2]; to this we add now the observation that when $B = \mathrm{Aut}(\mathcal{C})$ itself,

$$\mathrm{Aut}(\mathcal{C}){\blacktriangleright\!\!\!<}\mathrm{Aut}(\mathcal{C}){\cong}\mathrm{Aut}(\mathcal{C}){>\!\!\!\triangleleft}\mathrm{Aut}(\mathcal{C}),$$

i.e. in some sense should be viewed as the closest one may come to something in between like $\mathrm{Aut}(\mathcal{C}) \otimes \mathrm{Aut}(\mathcal{C})$, having an equal description either as a tensor product algebra with cross coalgebra or a tensor product coalgebra with a cross algebra. (We recall that there is no general tensor product of braided groups in a given braided category; the tensor product algebra and coalgebra themselves do not fit together due to 'tangling up' when the category is truly braided.) The construction can be iterated and leads to concrete R-matrix formulae for a n-fold 'spin chain' braided group $B(R){>\!\!\!\triangleleft}B(R){>\!\!\!\triangleleft}\cdots{>\!\!\!\triangleleft}B(R)$ where $B(R)$ are the braided matrices [7].

In Section 3 we solve the reconstruction problem for the forgetful functor

$$\mathcal{C}^{\circ} \to \mathcal{C}$$

where $\mathcal{C}^{\circ}$ is the dual[5][6] or 'centre' of a monoidal category. The solution is the braided group $\mathrm{Aut}(\mathcal{C}){\blacktriangleright\!\!\!<}\mathrm{Aut}(\mathcal{C})^*$, which we show is dual-quasitriangular. This makes it an example of some kind of braided codouble.

In Section 4 we make some remarks about double bosonisations and general braided double crossproducts, also a topic of recent interest. In fact, it was mentioned in the introduction of q-alg/9511001[12] that a theory of double cross products $B{\bowtie}C$ works fine in a braided category but does *not* include the general construction of the 'braided double' $B{\bowtie}B^*$ due to becoming 'tangled up'.

2 Reconstruction from $\mathcal{C}^B \to \mathcal{C}$

Let $\mathcal{C}$ be a braided category[16], with braiding $\Psi = \times$ and inverse braiding $\Psi^{-1} = \times$. We use the diagrammatic methods for braided groups due to the author in [2][3][4][13], where the product is represented as $\curlyvee$ etc. Contrary to recent misconceptions, such a notation for braided algebra is *not* in Yetter's fine paper[17] on crossed modules; $\curlyvee$ there refers to ordinary Hopf algebras in the category of vector spaces and $\times$ to a completely different braided category (so if one looks at the paper in detail, there is nothing like braided groups or braided algebra in [17]). We suppress the associativity $X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z$ for objects $X, Y, Z \in \mathcal{C}$ by Mac Lane's coherence theorem. We assume throughout that $\mathcal{C}$ is rigid, i.e. every object comes with a dual X^*, evaluation $\mathrm{ev} = \cup$ and coevaluation $\mathrm{coev} = \cap$. Here $\cap : \underline{1} \to X \otimes X^*$ but the unit object

[2]The coalgebra part of this reconstruction problem with $\mathcal{C} = \mathcal{M}^H$ was recently considered in the preprint of [15]. The full solution (the entire braided group structure) and the fact that it was already known from [4] was pointed out by the author in a letter to Pareigis during his preparation of the final version of [15].

for the tensor product is omitted in the diagrammatic notation. All categories are assumed equivalent to small ones.

We let $\mathcal{C}$ be such that the identity functor $i:\mathcal{C}\to\mathcal{C}$ obeys the representability condition for comodules, i.e. there exists an object $A\in\mathcal{C}$ such that

$$\theta:\mathrm{Mor}(A,V)\cong\mathrm{Nat}(i,i\otimes V)$$

for any object V in $\mathcal{C}$ and such that the induced maps

$$\theta_n:\mathrm{Mor}(A^n,V)\cong\mathrm{Nat}(i^n,i^n\otimes V)$$

are also isomorphisms. Here the induced maps θ_n defined by composing with the morphisms $\{\beta_X:X\to X\otimes A\}$ making up the natural transformation corresponding to $\mathrm{id}:A\to A$. In this situation one has [1] a braided group $A=\mathrm{Aut}(\mathcal{C})$ living in $\mathcal{C}$ and β_X is a *tautological coaction* of it on each object X. For example, we may suppose for convenience that $\mathcal{C}$ is rigid and cocomplete over itself. These same constructions go through more generally when, for example, $i:\mathcal{C}\to\bar{\mathcal{C}}$ where $\bar{\mathcal{C}}$ is a larger category (eg some cocompletion of $\mathcal{C}$), yielding $\mathrm{Aut}(\mathcal{C})\in\bar{\mathcal{C}}$). We use for that the more general reconstruction from a functor $F:\mathcal{C}\to\mathcal{V}$, which is defined similarly to the identity case above[3]; see [1][13][14].

Now let B be another braided group in $\mathcal{C}$. The category $\mathcal{C}^B$ of all braided comodules $(X,\blacktriangleleft)$ is monoidal[1], where $X\in\mathcal{C}$ and $\blacktriangleleft: X\to X\otimes B$ is the coaction.

Lemma 2.1 *In the above situation, the forgetful functor $F:\mathcal{C}^B\to\mathcal{C}$ satisfies the representability assumption with respect to the object* $\mathrm{Aut}(\mathcal{C})\otimes B$.

Proof We write $A=\mathrm{Aut}(\mathcal{C})$. Given any $\phi:A\otimes B\to V$ we define $\Theta(\phi)\in\mathrm{Nat}(F,F\otimes V)$ by $\Theta(\phi)_{(X,\blacktriangleleft)}=(\mathrm{id}\otimes\phi)(\beta_X\otimes\mathrm{id})\ \blacktriangleleft$. It is clearly a natural transformation since it can also be written as $\Theta(\theta^B)_X=\theta^B_X\circ\blacktriangleleft$, where $\theta^B\in\mathrm{Nat}(i\otimes B,i\otimes V)$ corresponds to ϕ via $\theta^B(\phi)_X=(\mathrm{id}\otimes\phi)\beta_X$. In the converse direction, given such a natural transformation Θ, we define $\theta^B(\Theta)_X=(\mathrm{id}\otimes\epsilon\otimes\mathrm{id})\circ\Theta_{(X\otimes B,\Delta)}$ as clearly a natural transformation in $\mathrm{Nat}(i\otimes B,i\otimes V)$, where we view $X\otimes B$ as in $\mathcal{C}^B$ by the trivial coaction on X and the coproduct on B. This then corresponds to a morphism $\phi:A\otimes B\to V$. These constructions are mutually inverse. Thus, given Θ we define θ^B as stated. Then

$$\Theta(\theta^B)_X=\theta^B_X\circ\blacktriangleleft=(\mathrm{id}\otimes\epsilon\otimes\mathrm{id})\Theta_{(X\otimes B,\Delta)}\circ\blacktriangleleft=(\mathrm{id}\otimes\epsilon\otimes\mathrm{id})(\blacktriangleleft\otimes\mathrm{id})\Theta_X=\Theta_X$$

since $\blacktriangleleft: X\to X\otimes B$ is a morphism $(X,\blacktriangleleft)\to(X\otimes B,\Delta)$ in $\mathcal{C}^B$ (due to $\blacktriangleleft$ a coaction) and Θ is natural. Conversely, given θ^B we define Θ as stated. Then

$$\theta^B(\Theta)_X=(\mathrm{id}\otimes\epsilon\otimes\mathrm{id})\Theta_{(X\otimes B,\Delta)}=(\mathrm{id}\otimes\epsilon\otimes\mathrm{id})\theta^B_{X\otimes B}(\mathrm{id}\otimes\Delta)=\theta^B_X$$

[3] And does *not* require either $\mathcal{C}$ or $\mathcal{V}$ to be both cocomplete and rigid, see p205-206 of [1]. One may take for example $\mathcal{V}$ cocomplete and the image of $\mathcal{C}$ rigid, as explained in [13], again long before [15].

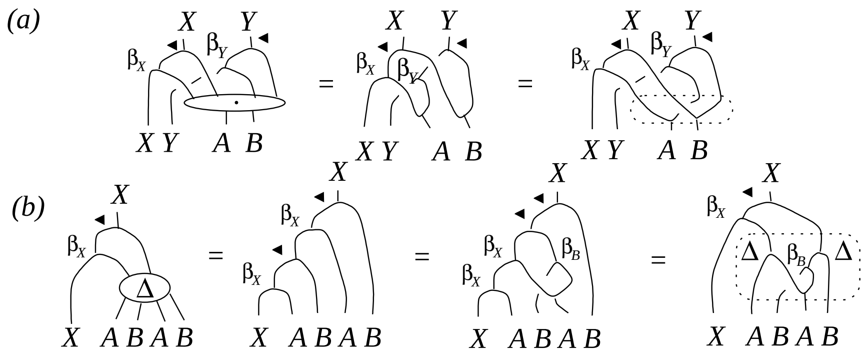

Figure 1: Proof of Theorem 2.2

since $\epsilon : X \otimes B \to X$ is a morphism in $\mathcal{C}$ and θ^B is natural. Similarly for the higher order representability conditions. $\square$

Theorem 2.2 *cf. [4] Let B be a braided group in $\mathcal{C}$. Braided reconstruction[1] from the forgetful functor $F : \mathcal{C}^B \to \mathcal{C}$ yields the* prebosonisation *braided group* $\mathrm{Aut}(\mathcal{C}){\blacktriangleright\!\!<}B$, *with the braided tensor product algebra and the cross coproduct coalgebra by the tautological coaction $\beta_B : B \to B \otimes \mathrm{Aut}(\mathcal{C})$ as an object of $\mathcal{C}$.*

Proof We routinely apply the reconstruction theorem as presented diagrammatically in [13][14]. The representing object is $A \otimes B$ from Lemma 2.1 and $\beta_{(X,\blacktriangleleft)} = (\beta_X \otimes \mathrm{id}) \circ \blacktriangleleft$ corresponds to the identity on $A \otimes B$. The product is defined in terms of this and $\beta_{(X \otimes Y, \blacktriangleleft \otimes \blacktriangleleft)}$, which is the middle box in Figure 1(a). From this we see that the product on $A \otimes B$ is the braided tensor product algebra (the dotted box). The coproduct is defined as such that $\beta_{(X,\blacktriangleleft)}$ is a coaction, see the first equality in Figure 1(b). The second equality is naturality under $\blacktriangleleft : X \to X \otimes B$ as a morphism in $\mathcal{C}$. We then use that $\blacktriangleleft$ is a coaction. This identifies the reconstructed coproduct as a cross coproduct by β_B (the dotted box). Cross product braided groups are in [4] and we turn that up-side-down. The result is a braided group due to the braided-commutativity of $\mathrm{Aut}(\mathcal{C})$[1]. $\square$

An example $BGL_q(2){\blacktriangleright\!\!<}A_q^2$, where $B = A_q^2$ is the quantum plane, is computed explicitly in [18]. Note that to obtain $BGL_q(2)$ (the braided group version of $GL_q(2)$) one uses the braided reconstruction in the slightly more general form where $\mathcal{C}$ consists of finite-dimensional objects and $\mathrm{Aut}(\mathcal{C})$ lives more precisely in its cocompletion[1].

Also, we can clearly iterate Theorem 2.2 to obtain braided groups

$$(\mathrm{Aut}(\mathcal{C}){\blacktriangleright\!\!<}(\mathrm{Aut}(\mathcal{C}){\blacktriangleright\!\!<}\cdots{\blacktriangleright\!\!<}B)\cdots)$$

or 'braided chain lattices'. Although there is in general no tensor product of braided groups, Theorem 2.2 says that we can always 'tensor' by $\mathrm{Aut}(\mathcal{C})$ in this way.

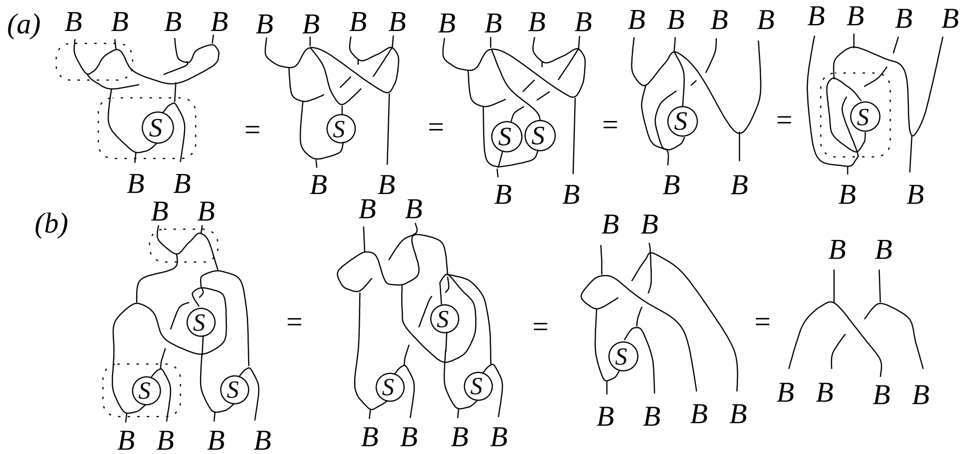

Figure 2: Proof of Lemma 2.3

Now we consider $B = \mathrm{Aut}(\mathcal{C})$ itself. By Theorem 2.2 we obtain a braided group $\mathrm{Aut}(\mathcal{C}){\blacktriangleright\!\!<}\mathrm{Aut}(\mathcal{C})$, which we call the *square* of $\mathrm{Aut}(\mathcal{C})$.

Lemma 2.3 *Let B be any braided group. Then $B{\blacktriangleright\!\!<}B$ by the braided adjoint coaction (and braided tensor algebra) is isomorphic to $B{>\!\!\!\triangleleft}B$ by the braided adjoint action (and braided tensor coalgebra).*

Proof In general, neither of these will be braided groups – so this is an isomorphism of algebras and coalgebras. The former is shown in Figure 2(a). We apply the isomorphism (the upper dotted box), then the braided tensor product algebra, then the inverse of the isomorphism (lower dotted box). We use the coproduct homomorphism property, antipode antimultiplicativity, the antipode axiom to cancel a loop, and finally recognise the cross product by the adjoint action (dotted box on the right). Figure 2(b) does the computation for the coproduct; we apply the isomorphism (dotted box), the cross coproduct by the braided adjoint action, and then the inverse of the isomorphism (dotted box). We use the coproduct homomorphism property and cancel two resulting antipode loops. Cancelling the resulting antipode loop gives the braided tensor coproduct on the right. $\square$

Proposition 2.4 *Reconstruction from the forgetful functor $\mathcal{C}^{\mathrm{Aut}(\mathcal{C})} \to \mathcal{C}$ yields*

$$\mathrm{Aut}(\mathcal{C}){\blacktriangleright\!\!<}\mathrm{Aut}(\mathcal{C}) \cong \mathrm{Aut}(\mathcal{C}){>\!\!\!\triangleleft}\mathrm{Aut}(\mathcal{C})$$

where the left hand side is a cross coproduct by the adjoint coaction (and braided tensor product algebra) and the left hand side is the cross product[4] by the adjoint action[10] (and braided tensor product coalgebra).

Proof From the explicit realisation as a coend[1] [19] $\mathrm{Aut}(\mathcal{C}) = \int_X X^* \otimes X$ it is clear that the canonical coaction $\beta_{\mathrm{Aut}(\mathcal{C})}$ is the braided adjoint coaction [13] of any braided group B on itself. Indeed, the coaction $\mathrm{id} \otimes \beta_X$ on each X coincides with the coproduct $\mathrm{Aut}(\mathcal{C}) \to \mathrm{Aut}(\mathcal{C}) \otimes \mathrm{Aut}(\mathcal{C})$ as part of the reconstruction, the coaction on X^* is conjugate to this via the antipode S. We then use Lemma 2.3. $\square$

One knows from [11] that any trivial principal bundle has a cross product form, and Proposition 2.4 tell us that in the present case it is $\mathrm{Aut}(\mathcal{C}) \bowtie \mathrm{Aut}(\mathcal{C})$. This is then explicitly a trivial braided principal bundle with right coaction $\mathrm{id} \otimes \Delta$ and the canonical inclusion of the left hand $\mathrm{Aut}(\mathcal{C})$ as 'base' of the bundle. It also means that the Hopf algebra is as close as one can come to $\mathrm{Aut}(\mathcal{C}) \underline{\otimes} \mathrm{Aut}(\mathcal{C})$ as a braided group.

A concrete example is $BG_q \blacktriangleright\!\!< BG_q$ where the braided coordinate rings BG_q are quotients of the braided matrices[7] $B(R)$. Writing the matrix generators of the first copy as $\mathbf{u}$ and the second as $\mathbf{v}$, their relations and that of the braided tensor product (braid statistics) between them are

$$R_{21}\mathbf{u}_1 R\mathbf{u}_2 = \mathbf{u}_2 R_{21}\mathbf{u}_1 R, \quad R_{21}\mathbf{v}_1 R\mathbf{v}_2 = \mathbf{v}_2 R_{21}\mathbf{v}_1 R, \quad R^{-1}\mathbf{v}_1 R\mathbf{u}_2 = \mathbf{u}_2 R^{-1}\mathbf{v}_1 R.$$

The third relations here correspond to the braiding[7]

$$\Psi(R^{-1}\mathbf{u}_1 \otimes R\mathbf{u}_2) = \mathbf{u}_2 R^{-1} \otimes \mathbf{u}_1 R$$

(written in [7] with all R to one side) between any two independent copies of $B(R)$. By Theorem 2.2, this $BG_q \blacktriangleright\!\!< BG_q$ is a braided group with

$$\Delta\mathbf{u} = \mathbf{u} \otimes \mathbf{u}, \quad \Delta\mathbf{v} = \mathbf{u}^{-1} \bullet \mathbf{v} \otimes \mathbf{u}\mathbf{v}$$

where $\mathbf{u}^{-1}$ is to be exchanged with $\mathbf{v}$ using the relations $R^{-1}\mathbf{u}_1 \bullet R\mathbf{v}_2 = \mathbf{v}_2 R^{-1} \bullet \mathbf{u}_1 R$ and multiplied with $\mathbf{u}$ (these are the relations of the braided tensor product of the copy generated by $\mathbf{v}$ with that generated by $\mathbf{u}$, as a way of describing the braided adjoint coaction as conjugation[8]).

By Proposition 2.4, this is isomorphic as a braided group to $BG_q \bowtie BG_q$, with relations

$$R_{21}\mathbf{u}_1 R\mathbf{u}_2 = \mathbf{u}_2 R_{21}\mathbf{u}_1 R, \quad R_{21}\mathbf{v}_1 R\mathbf{v}_2 = \mathbf{v}_2 R_{21}\mathbf{v}_1 R, \quad R_{21}\mathbf{v}_1 R\mathbf{u}_2 = \mathbf{u}_2 R_{21}\mathbf{v}_1 R$$

and the coproduct $\Delta\mathbf{u} = \mathbf{u} \otimes \mathbf{u}$, $\Delta\mathbf{v} = \mathbf{v} \otimes \mathbf{v}$. Also, since $B(R)$ is also the braided enveloping bialgebra $U(\mathcal{L})$, where $\mathcal{L}$ is the braided Lie algebra associated to the R-matrix[10], we have a braided group $BG_q \bowtie U(\mathcal{L})$, which can be viewed as the algebra of observables of a braided particle with 'generalised momentum' $\mathcal{L}$ moving under the adjoint action on the braided space BG_q. The braided-Lie bracket is the adjoint action and we use the known R-matrix formulae for that to obtain

$$\mathbf{v}_1 R\mathbf{u}_2 = \cdot \circ \mathbf{v}_1 \triangleright \Psi(\mathbf{v}_1 \otimes R\mathbf{u}_2) = [\mathbf{v}_1, R\mathbf{u}_2] R^{-1}\mathbf{v}_1 R = R_{21}^{-1}\mathbf{u}_2 R_{21}\mathbf{v}_1 R$$

to derive the cross product as stated above. This braided group $BG_q{>\!\!\!\triangleleft} U(\mathcal{L})$ is related by transmutation to the quantum double as explained in [9]. See also the next section.

These formulae also work fine at the braided bialgebra level $B(R){>\!\!\!\triangleleft} B(R)$ for any biinvertible R-matrix, as one may verfiy directly. And in spite of its origin as a braided cross product, the formulae are remarkably symmetric between the two copies of $B(R)$, reflecting the role as 'tensor product'. Moreover, the construction can be iterated to n copies of BG_q or $B(R)$, i.e. a 'braided spin chain' $B(R){>\!\!\!\triangleleft} B(R){>\!\!\!\triangleleft}\cdots{>\!\!\!\triangleleft} B(R)$. This is generated by $\mathbf{u}^{(i)}$ (one for each copy of $B(R)$) with relations

$$R_{21}\mathbf{u}_1^{(i)}R\mathbf{u}_2^{(j)}=\mathbf{u}_2^{(j)}R_{21}\mathbf{u}_1^{(i)}R,\quad \forall i\ge j$$

and coproduct $\Delta\mathbf{u}^{(i)}=\mathbf{u}^{(i)}\otimes\mathbf{u}^{(i)}$, forming a braided group with braiding Ψ as above between any two copies of $B(R)$. This braided group is related by transmutation to the iterated double crossproducts $A(R)\bowtie\cdots\bowtie A(R)$ in [20]. It would be very interesting to relate this approach also to the multiloop braided algebras in [21].

3 Reconstruction from $\mathcal{C}^\circ$

Given a monoidal category $\mathcal{C}$ one has a dual monoidal category $\mathcal{C}^\circ$[5][6]. It also arose at about the same time as a 'centre' or 'double' construction, see [14]. Objects of $\mathcal{C}^\circ$ are pairs (V,λ) where V is an object of $\mathcal{C}$ and $\lambda_X:V\otimes X\to X\otimes V$ is a natural transformation $\lambda\in\mathrm{Nat}(V\otimes i,i\otimes V)$ which 'represents' the tensor product of $\mathcal{C}$ in the sense

$$\lambda_{\underline{1}}=\mathrm{id},\quad \lambda_{X\otimes Y}=\lambda_Y\circ\lambda_X.$$

Morphisms are morphisms of $\mathcal{C}$ intertwining the corresponding λ. Actually (an observation due to Drinfeld) $\mathcal{C}^\circ$ in the present case is braided, with $\Psi_{(V,\lambda),(W,\mu)}=\lambda_W$. In our present applications the category $\mathcal{C}$ is itself braided as well.

Proposition 3.1 *Reconstruction from the forgetful functor $\mathcal{C}^\circ\to\mathcal{C}$ yields the dual-quasitriangular braided group $\mathrm{Aut}(\mathcal{C}){\blacktriangleright\!\!\!<}\mathrm{Aut}(\mathcal{C})^*$, a cross coproduct by the braided coadjoint coaction.*

Proof From [6] we know that $\mathcal{C}^\circ\cong\mathcal{C}_{\mathrm{Aut}(\mathcal{C})}$, the category of right $\mathrm{Aut}(\mathcal{C})$-modules, which is essentially the same thing as $\mathrm{Aut}(\mathcal{C})^*$-comodules (we assume that a suitable dual braided group exists). Then we can apply Theorem 2.2 with $B=\mathrm{Aut}(\mathcal{C})^*$ to obtain $\mathrm{Aut}(\mathcal{C}){\blacktriangleright\!\!\!<}\mathrm{Aut}(\mathcal{C})^*$ as the reconstructed braided group. There is also a second 'opposite' product defined in Figure 3(a) via $\beta_{(Y\otimes X,\blacktriangleleft\otimes\blacktriangleleft)}$. We then use that $A=\mathrm{Aut}(\mathcal{C})$ itself is braided-commutative. If A^* has an opposite product itself characterised by Figure 3(b) for all X, then we see that $(A{\blacktriangleright\!\!\!<}A^*)^{\mathrm{op}}=A\underline{\otimes}A^{*\mathrm{op}}$ (the braided tensor product). Finally in the braided reconstruction theory there is a dual-quasitriangular structure defined in Figure 3(c) by the braiding of $\mathcal{C}^\circ$, which we write with λ

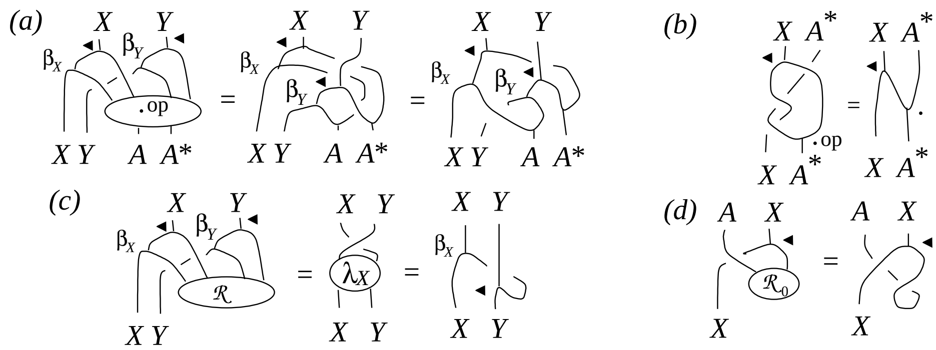

Figure 3: Proof of Proposition 3.1

in terms of the corresponding coaction ◀ according to [6]. We see that if there is morphism $\mathcal{R}_0 : A \otimes A^* \to \underline{1}$ obeying Figure 3(d) then $\mathcal{R} = \mathcal{R}_0 \circ (\mathrm{id} \otimes \epsilon \otimes \epsilon \otimes \mathrm{id})$. □

This braided group $\mathrm{Aut}(\mathcal{C}) \blacktriangleright\!\!< \mathrm{Aut}(\mathcal{C})^*$ is therefore some kind of 'braided codouble' of $\mathrm{Aut}(\mathcal{C})$ in spite of the the fact that in general in a braided category this does not exist (see the next section). Explicit examples are similar to those in the preceding section since BG_q is essentially self-dual.

4 Braided double cross products

This section is a kind of appendix to [12]. It was mentioned in its introduction:

"the double cross product $B \bowtie C$ construction does go through in a braided category, but the key example of a general braided double $B \bowtie B^{*\mathrm{op}}$ does not"

(a paraphrase) – but details of the braided double cross product were left unpublished due to this basic lack of examples. Since that work, there has nevertheless been a lot of interest in braided versions and generalisations of double cross products and bicrossproducts[22][23], and for this reason we would like to publish now our calculations[24] mentioned in [12]. They can by now be viewed as a special case of the general constructions in [22][23], but a case important enough to study directly as we do now.

Proposition 4.1 *[24] A* matched pair *of braided groups is* $(B, C, \triangleright, \triangleleft)$ *where* B, C *are braided groups in a category* $\mathcal{C}$, $\triangleright$ *makes* B *a braided left* C*-module coalgebra[13],* $\triangleleft$ *makes* C *a braided right* B*-module coalgebra and* $\triangleright, \triangleleft$ *obey the conditions in Figure 4(a1)-(a3) (and are trivial acting on 1). In this case there is a* double cross product *braided group* $B \bowtie C$ *with product the dotted box in Figure 4(b) and the tensor product coproduct.*

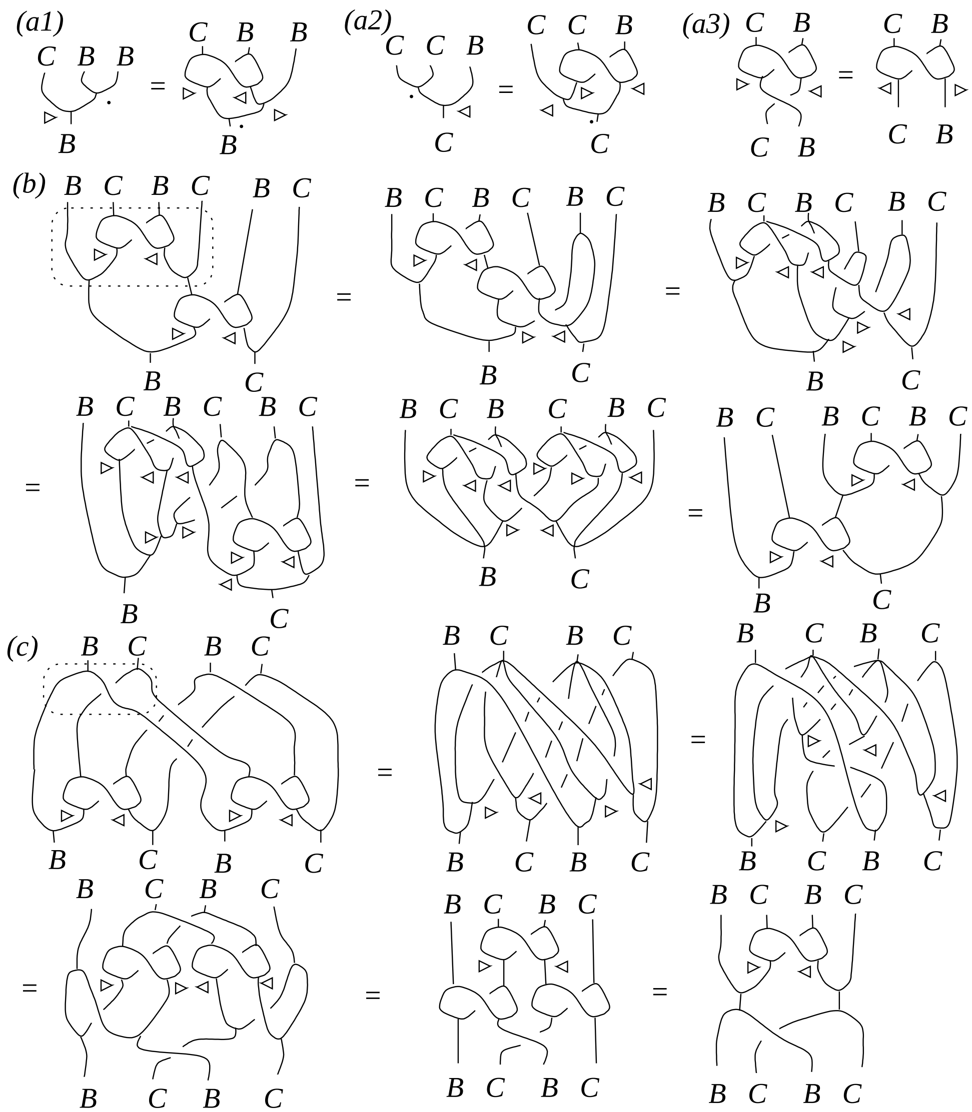

Figure 4: Proof of Proposition 4.1

Proof Assuming a matched pair, Figure 4(b) checks that the braided double cross product (the dotted box) is associative. We use the coproduct homomorphism property for the first equality. The second equality is that $\triangleleft$ respects the coproduct of C and that $\triangleright$ is a left action. The third equality is axiom (a2). The fourth is a reorganisation and associativity of B, C to obtain an expression which is symmetric under mirror-reflection (followed by reversal of braid crossings). Therefore by the mirror image of the above steps, taken in the reverse order, we obtain the final expression. Part (c) checks that the braided tensor coproduct (the dotted box) obeys the homomorphism property. The second equality is axiom (a3). The third equality is another reorganisation. The fourth then uses that $\triangleright, \triangleleft$ respect coproducts. Finally we use the coproduct homomorphism properties of B, C. $\square$

Conversely, given a braided group X factorising into braided groups B, C, one may recover $\triangleright, \triangleleft$ and $X \cong B \bowtie C$ by a similar proof to the unbraided case in [14].

As a possible example, we might try to build $D(B) = B \bowtie B^{*\mathrm{op}}$ by braided coadjoint actions. The required coadjoint actions indeed exist but axiom (a3) in Figure 4 fails due to 'tangling up'. There is no such problem in a *symmetric* monoidal category, however. In this case, when $\mathcal{C} = {}_H\mathcal{M}$ is the modules over a triangular Hopf algebra, one obtains the bosonisation of this symmetric-category double as

$$(B \bowtie B^{*\mathrm{op}}) >\!\!\!\triangleleft H \cong B >\!\!\!\triangleleft H \triangleright\!\!\!< B^{*\mathrm{op}},$$

the double-bosonisation. This is explained in [25] (in the super case) and was indeed one of the main motivations behind the double-bosonisation construction in [12]; in general $B \bowtie B^{*\mathrm{op}}$ does not exist but the double-bosonisation does!

References

[1] S. Majid. Braided groups. *J. Pure and Applied Algebra*, 86:187–221, 1993.

[2] S. Majid. Braided groups and algebraic quantum field theories. *Lett. Math. Phys.*, 22:167–176, 1991.

[3] S. Majid. Transmutation theory and rank for quantum braided groups. *Math. Proc. Camb. Phil. Soc.*, 113:45–70, 1993.

[4] S. Majid. Cross products by braided groups and bosonization. *J. Algebra*, 163:165–190, 1994.

[5] S. Majid. Representations, duals and quantum doubles of monoidal categories. *Suppl. Rend. Circ. Mat. Palermo, Ser. II*, 26:197–206, 1991.

[6] S. Majid. Braided groups and duals of monoidal categories. *Canad. Math. Soc. Conf. Proc.*, 13:329–343, 1992.

[7] S. Majid. Examples of braided groups and braided matrices. *J. Math. Phys.*, 32:3246–3253, 1991.

[8] S. Majid. Quantum and braided linear algebra. *J. Math. Phys.*, 34:1176–1196, 1993.

[9] S. Majid. Braided matrix structure of the Sklyanin algebra and of the quantum Lorentz group. *Commun. Math. Phys.* 156:607–638, 1993.

[10] S. Majid. Quantum and braided Lie algebras. *J. Geom. Phys.*, 13:307–356, 1994.

[11] S. Majid. Diagrammatics of braided group gauge theory. *Preprint*, 1996.

[12] S. Majid. Double bosonisation and the construction of $U_q(g)$, 1995. To appear in *Math. Proc. Camb. Phil. Soc.*

[13] S. Majid. Algebras and Hopf algebras in braided categories. volume 158 of *Lec. Notes in Pure and Appl. Math*, pages 55–105. Marcel Dekker, 1994.

[14] S. Majid. *Foundations of Quantum Group Theory.* Cambridge Univeristy Press, 1995.

[15] B. Pareigis. Reconstruction of hidden symmetries. *J. Algebra*, 183:90–154, 1996.

[16] A. Joyal and R. Street. Braided monoidal categories. Macquarie Report, 1986.

[17] D.N. Yetter. Quantum groups and representations of monoidal categories. *Math. Proc. Camb. Phil. Soc.*, 108:261–290, 1990.

[18] S. Majid. Some comments on bosonisation and biproducts. *Czech J. Phys.*, 47:151–171, 1997.

[19] V.V. Lyubashenko. Tangles and Hopf algebras in braided categories. *J. Pure and Applied Algebra*, 98:245–278, 1995.

[20] S. Majid. More examples of bicrossproduct and double cross product Hopf algebras. *Isr. J. Math*, 72:133–148, 1990.

[21] F. Nill. Structure of monodromy algebras and Drinfeld doubles. *Rev.Math.Phys.*, 9:371-395, 1997.

[22] Yu. N. Bespalov and B. Drabant. Cross product bialgebras, I. *Preprint*, Damtp/98-9, February, 1998.

[23] S. Zhang and H.-X. Chen. The double biproduct in braided tensor categories. *Preprint*, Fudan/Nanjing Univ., 1998.

[24] S. Majid. Notes to ref. [12], *Unpublished*, 1995.

[25] S. Majid. New quantum groups by double-bosonisation. *Czech J. Phys.*, 47:79–90, 1997.